

Kähler-Einstein metrics and compactifications of \mathbb{C}^n

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1 Introduction and statements of the main results

The study of holomorphic and isometric immersions of a Kähler manifold (M, g) into a Kähler manifold (N, G) started with Calabi. In his famous paper [3] he considered the case when the ambient space (N, G) is a complex space form, i.e. its holomorphic sectional curvature K_N is constant.

There are three types of complex space forms : *flat, hyperbolic or elliptic* according as the holomorphic sectional curvature is *zero, negative, or positive*. Every n -dimensional complex space form is, after multiplying by a suitable constant, locally holomorphically isometric to one of the following: the complex euclidean space \mathbb{C}^n with the flat metric; the unit ball in \mathbb{C}^n with its Bergman metric of negative holomorphic sectional curvature; the n -dimensional complex projective space $\mathbb{C}P^n$ endowed with the Fubini-Study metric g_{FS} of positive holomorphic sectional curvature. Recall that g_{FS} is the metric whose associated Kähler form is given by

$$\omega_{FS} = \frac{i}{2} \partial \bar{\partial} \log \sum_{j=0}^n |Z_j|^2 \quad (1)$$

for a homogeneous coordinate system $[Z_0, \dots, Z_n]$.

One of Calabi’s main results asserts that if a complex space form can be holomorphically and isometrically immersed into another complex space form then their sectional holomorphic curvatures have the same sign.

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The natural and interesting extension of this theorem is the case when the metric g is Kähler-Einstein (K-E from now on) and (N, G) is a complex space form. The first result in this direction is due to Chern [5] who showed that if M is a hypersurface immersed in N then M is totally geodesic if $K_N \leq 0$, while if $K_N > 0$ then either M is totally geodesic or a quadric.

These results were further generalized by Umehara [17] and Tsukada [16]. The first proved that, if $K_N \leq 0$, and (M, g) is K-E manifold holomorphically and isometrically immersed in N then it is totally geodesic. The second showed that, if $K_N > 0$, and the codimension of M in N is 2, then M is totally geodesic or a quadric. A systematic study for general codimension, in the case $K_N > 0$, is due to Hulin in [8] and [9]. Among other things she showed that if (M, g) is a K-E embedded submanifold of $(\mathbb{C}P^n, g_{FS})$ then its scalar curvature is strictly positive.

The aim of this paper is to extend Hulin's proof so to generalize her result to embedded compact K-E submanifolds of projectively algebraic Kähler manifolds of (N, G) which share same holomorphic and isometric properties with $(\mathbb{C}P^n, g_{FS})$. More precisely we assume throughout this paper that N contains an analytic subvariety Y , not necessarily smooth, such that $X = N \setminus Y$ is biholomorphic to \mathbb{C}^n . These spaces are called *compactifications of \mathbb{C}^n* and they have been extensively studied being the object of a famous problem posed by Hirzebruch [7]. Of course $\mathbb{C}P^n$ is an example of these spaces, since $Y = \mathbb{C}P^{n-1}$ satisfies all the required properties.

As far as the Kähler metric G is concerned, we impose three conditions which are clearly satisfied by the Fubini-Study metric g_{FS} on $\mathbb{C}P^n$ (see Section 2).

Condition 1: G is real analytic. This condition, which is clearly very natural in this setting, has been extensively studied by Calabi [3]. The ingenious idea of Calabi was the introduction, in a neighborhood of a point $p \in N$ of a very special Kähler potential D_p for the metric G , which he christened *diastasis*. Recall that a Kähler potential is a analytic function Φ defined in a neighborhood of a point p such that $\Omega = \frac{i}{2} \bar{\partial} \partial \Phi$, where Ω is the Kähler form associated to G . In a complex coordinate system (Z) around p

$$G_{\alpha\beta} = 2G\left(\frac{\partial}{\partial Z_\alpha}, \frac{\partial}{\partial Z_\beta}\right) = \frac{\partial^2 \Phi}{\partial Z_\alpha \partial \bar{Z}_\beta}.$$

A Kähler potential is not unique: it is defined up to the sum with the real part of a holomorphic function. By duplicating the variables Z and \bar{Z} a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined

in a neighborhood U of the diagonal containing $(p, \bar{p}) \in N \times \bar{N}$ (here \bar{N} denotes the manifold conjugated of N). The *diastasis function* is the Kähler potential D_p around p defined by

$$D_p(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Among all the potentials the diastasis is characterized by the fact that in every coordinates system (Z) centered in p

$$D_p(Z, \bar{Z}) = \sum_{|j|, |k| \geq 0} a_{jk} Z^j \bar{Z}^k,$$

with $a_{j0} = a_{0j} = 0$ for all multi-indices j . The following proposition shows the importance of the diastasis in the context of holomorphic maps between Kähler manifolds.

Proposition 1.1 (Calabi) *Let $\varphi : (M, g) \rightarrow (N, G)$ be a holomorphic and isometric embedding between Kähler manifolds and suppose that G is real analytic. Then g is real analytic and for every point $p \in M$*

$$\varphi(D_p) = D_{\varphi(p)},$$

where D_p (resp. $D_{\varphi(p)}$) is the diastasis of g relative to p (resp. of G relative to $\varphi(p)$).

We are now in the position to state the second condition

Condition A: there exists a point $p_ \in X = N \setminus Y$ such that the diastasis D_{p_*} is globally defined and non-negative on X .*

In order to describe the third condition we need to introduce the concept of Bochner's coordinates (cfr. [1], [3], [8], [9], [15], [13]). Given a real analytic Kähler metric G on N and a point $p \in N$, one can always find local (complex) coordinates in a neighborhood of p such that

$$D_p(Z, \bar{Z}) = |Z|^2 + \sum_{|j|, |k| \geq 2} b_{jk} Z^j \bar{Z}^k,$$

where D_p is the diastasis relative to p . These coordinates, uniquely defined up to a unitary transformation, are called *the Bochner's coordinates* with respect to the point p .

One important feature of this coordinates which we are going to use in the proof of our main theorem is the following:

Theorem 1.2 (Calabi) *Let $\varphi : (M, g) \rightarrow (N, G)$ be a holomorphic and isometric embedding between Kähler manifolds and suppose that G is real analytic. If (z_1, \dots, z_m) is a system of Bochner's coordinates in a neighborhood U of $p \in M$ then there exists a system of Bochner's coordinates (Z_1, \dots, Z_n) with respect to $\varphi(p)$ such that*

$$Z_1|_{\varphi(U)} = z_1, \dots, Z_m|_{\varphi(U)} = z_m. \quad (2)$$

Condition B: the Bochner's coordinates with respect to the point $p_ \in X$, given by the previous condition (A), are globally defined on X .*

Once that the previous conditions 1, (A) and (B) are satisfied, the obstruction to the existence of K-E submanifolds M of non-positive scalar curvature comes from comparing the volume of M with the “euclidean” volume of the “affine part” $\varphi(M) \cap X$, which, as we will see, is forced to be infinite.

Our first result is then the following:

Theorem 1.3 *Let N be a smooth projective compactification of X such that X is algebraically biholomorphic to \mathbb{C}^n and let G be a real analytic Kähler metric on N such that the following two conditions are satisfied:*

- (A) *there exists a point $p_* \in X$ such that the diastasis D_{p_*} is globally defined and non-negative on X ;*
- (B) *the Bochner's coordinates with respect to p_* are globally defined on X .*

Then any K-E sub-manifold (M, g) of (N, G) has positive scalar curvature.

It would be interesting to drop the assumption that X is algebraically biholomorphic to \mathbb{C}^n so to apply the previous result to any compactification of \mathbb{C}^n .

The algebraicity of the equivalence between X and \mathbb{C}^n comes in our argument only to estimate the euclidean volume of the affine part of M . This leads to the following problem, which we believe to be very interesting on its own sake:

Problem 1 *Let (N, Y) be a compactification of \mathbb{C}^n , M an analytic subvariety of pure dimension m , and $\pi: M \setminus (M \cap Y) \rightarrow \mathbb{C}^m$ an open projection. Is $\text{vol}(\pi(M \setminus (M \cap Y)))$ infinite? Or even stronger, is the complement of $\pi(M \setminus (M \cap Y))$ an analytic subvariety?*

When π is algebraic and N is projective, this is a well known result of Chevalley (see Theorem 13.2 in [2]). On the other hand, it is worthwhile pointing out that all known examples (even the non-projective ones!) satisfy our algebraicity assumption. In fact, it has been conjectured by Peternell-Schneider (Conjecture 5 in [12]) that this is always true.

About the smoothness assumption of N , we should remark that for $n \leq 3$ smooth implies projective (see [12]). It is also important to observe that if also Y were smooth in N then N would be the projective space and Y a linear subspace for $n \leq 6$ (and conjecturally for any n , see [12]).

As an application to the theory of compactification of \mathbb{C}^n , we point out the following:

Theorem 1.4 *Let N be a smooth compactification of X endowed with a real analytic K -E metric G . Suppose that the following two conditions are satisfied:*

- (A) *there exists a point $p_* \in X$ such that the diastasis D_{p_*} is globally defined and non-negative on X ;*
- (B1) *the Bochner's coordinates (Z_1, \dots, Z_n) around p_* extend to holomorphic functions (f_1, \dots, f_n) on X .*

Then $c_1(N) > 0$.

Observe that in the previous theorem we do not suppose as in Theorem 1.3 that N is projective and that X is biholomorphic to \mathbb{C}^n in an algebraic way. Moreover condition (B1) is weaker than condition (B) in Theorem 1.3 since the holomorphic functions (f_1, \dots, f_n) are not necessarily coordinates on X .

One should compare the previous theorem with the following theorem of Kodaira which goes deeply to the heart of compactifications of \mathbb{C}^n (see [10] and [12]):

Theorem 1.5 (Kodaira) *Let (N, Y) be a smooth projective compactification of \mathbb{C}^n with irreducible Y . Then N is a Fano manifold, i.e. $c_1(N) > 0$.*

The argument of the proofs of Theorems 1.3 and 1.4 give in fact the following:

Theorem 1.6 *Let N be a complex manifold equipped with a real analytic K -E metric G . Let $p_* \in N$ and let U be a neighborhood of p_* endowed with Bochner's coordinates (Z_1, \dots, Z_n) . Suppose that the following two conditions are satisfied:*

(A1) the diastasis D_{p_*} is non-negative on U ;

(B2) the Euclidean volume of U with respect to the Bochner's coordinates is infinite namely:

$$\int_U \frac{i^n}{2^n} dZ_1 \wedge d\bar{Z}_1 \wedge \dots \wedge dZ_n \wedge d\bar{Z}_n = \infty.$$

Then $c_1(N) > 0$.

In Section 2 we will show how condition (A) is related to geometric properties of a class of metrics, called polarized metrics, studied by Tian [15]. On the other hand conditions (B), (B1) and (B2) seem more difficult to be treated. For example we will see that either the flat torus either a Riemann surface of genus $g \geq 2$ endowed with the hyperbolic metric satisfy all the conditions of Theorem 1.6 on a open dense subset except condition (B2) and they have non-positive first Chern class.

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2 Comments on the assumptions

The easiest example of compactification of \mathbb{C}^n which satisfies condition (A) is given by $\mathbb{C}P^n = \mathbb{C}^n \cup Y$ endowed with the Fubini–Study metric g_{FS} , where $Y = \mathbb{C}P^{n-1}$ is the hyperplane $Z_0 = 0$. Indeed the diastasis with respect to $p_* = [1, 0, \dots, 0]$ is given by:

$$D_{p_*}(u, \bar{u}) = \log(1 + \sum_{j=1}^n |u_j|^2).$$

where (u_1, \dots, u_n) are the affine coordinates, namely $u_j = \frac{Z_j}{Z_0}, j = 1, \dots, n$.

Another example of compactification of \mathbb{C}^n satisfying condition (A) is given by the product of two complex projective spaces $N = \mathbb{C}P^r \times \mathbb{C}P^s, r + s = n$ endowed with the sum of the Fubini–Study metrics on $\mathbb{C}P^r$ and $\mathbb{C}P^s$ respectively (what follows obviously works for a product of finitely many projective spaces). In this case we take as $Y = (\mathbb{C}P^{r-1} \times \mathbb{C}P^s) \cup (\mathbb{C}P^r \times \mathbb{C}P^{s-1})$ where $\mathbb{C}P^{r-1} = \{Z_0 = 0\}$ and $\mathbb{C}P^{s-1} = \{W_0 = 0\}$. Here we are

equipping N with the homogeneous coordinates $[Z_0, \dots, Z_r] \times [W_0, \dots, W_s]$. The diastasis with respect to the point $p_* = [1, \dots, 0] \times [1, \dots, 0]$ is given by

$$D_{p_*}(u, \bar{u}) = \log(1 + \sum_{l=1}^{r+s} |u_l|^2),$$

where $u_j = \frac{Z_j}{Z_0}, j = 1, \dots, r$ and $u_{r+k} = \frac{W_k}{W_0}, k = 1, \dots, s$.

We also remark that, by Proposition 1.1, condition (A) is satisfied also by all the Kähler submanifolds of the previous examples.

We now generalize the previous three examples to a broader class of Kähler metrics called *polarized*. To describe this let L be a holomorphic line bundle on a compact complex manifold N . A Kähler metric on N is *polarized* with respect to L if the Kähler form Ω associated to G represents the Chern class $c_1(L)$ of L . The line bundle L is called a *polarization* of (N, G) . In terms of cohomology classes, a Kähler manifold (N, G) admits a polarization if and only if Ω is integral, i.e. its cohomology class $[\Omega]_{dR}$ in the de Rham group, is in the image of the natural map $H^2(M, \mathbb{Z}) \hookrightarrow H^2(M, \mathbb{C})$. The integrality of Ω implies, by a well-known theorem of Kodaira, that N is a projective algebraic manifold. This means that N admits a holomorphic embedding into some complex projective space $\mathbb{C}P^N$. In this case a polarization L of (N, G) is given by the restriction to N of the hyperplane line bundle on $\mathbb{C}P^N$. Given any polarized Kähler metric G on N one can find a hermitian metric h on L with its Ricci curvature form $\text{Ric}(h) = \Omega$ (see Lemma 1.1 in [15]).

For each positive integer k , we denote by $L^{\otimes k}$ the k -th tensor power of L . It is a polarization of the Kähler metric kG and the hermitian metric h induces a natural hermitian h^k metric on $L^{\otimes k}$ such that $\text{Ric}(h^k) = k\Omega$.

Denote by $H^0(N, L^{\otimes k})$ the space of global holomorphic sections of $L^{\otimes k}$. It is in a natural way a complex Hilbert space with respect to the norm

$$\|s\|_{h^k} = \langle s, s \rangle_{h^k} = \int_N h^k(s(x), s(x)) \frac{\Omega^n(x)}{n!} < \infty,$$

for $s \in H^0(N, L^{\otimes k})$.

For sufficiently large k we can define a holomorphic embedding of N into a complex projective space as follows. Let (s_0, \dots, s_{d_k}) be an orthonormal basis for $H^0(N, L^{\otimes k})$ and let $\sigma : U \rightarrow L^{\otimes k}$ be a trivialising holomorphic section on the open set $U \subset N$. Define the map

$$\varphi_\sigma : U \rightarrow \mathbb{C}^{d_k+1} \setminus \{0\}, x \mapsto \left(\frac{s_0(x)}{\sigma(x)}, \dots, \frac{s_{d_k}(x)}{\sigma(x)} \right). \quad (3)$$

If $\tau : V \rightarrow L$ is another holomorphic trivialisation then there exists a non-vanishing holomorphic function f on $U \cap V$ such that $\sigma(x) = f(x)\tau(x)$. Therefore one can define a holomorphic map

$$\varphi_k : N \rightarrow \mathbb{C}P^{d_k}, \quad (4)$$

whose local expression in the open set U is given by (3). It follows by the above mentioned Theorem of Kodaira that, for k sufficiently large, the map φ_k is an embedding into $\mathbb{C}P^{d_k}$ (see, e.g. [6] for a proof).

The Fubini–Study metric on $\mathbb{C}P^{d_k}$, which we denote by g_{FS}^k , restricts to a Kähler metric $G_k = \varphi_k^* g_{FS}^k$ on N which is polarized with respect to $L^{\otimes k}$. In [15] Tian christened the set of normalized metrics $\frac{1}{k}G_k$ as the *Bergmann* metrics on N with respect to L and he proves that the sequence $\frac{1}{k}G_k$ converges to the metric G in the C^2 -topology (see Theorem A in [15]). This theorem was further generalized in the C^∞ case by Ruan [13] (see also [11] and [18]).

The mentioned results have found important applications to the existence problem of K–E metrics. We now show that they also imply interesting conditions on the diastasis of any polarized Kähler metric.

Proposition 2.1 *Let N be a complex manifold endowed with a real analytic Kähler metric G polarized with respect to the line bundle L . Let (s_0, \dots, s_d) be a orthonormal basis for $H^0(M, L)$ and let $D = \{s_0 = 0\}$ be a divisor associated to L . Suppose that the following two conditions hold:*

- (C) *there exists a point $p_* \in N \setminus D$ such that $s_j(p_*) = 0, \forall j = 1, \dots, d$;*
- (D) *there exists a natural number k_0 such that for all $k \geq k_0$ $s_0^{\otimes k}$ is L^2 -orthogonal to $s_0^{\otimes j_0} \otimes \dots \otimes s_d^{\otimes j_d}$, $j_0 + \dots + j_d = k, j_0 \neq k$ with respect to the L^2 -product induced by h^k .*

Then D_{p_} , the diastasis of G with respect to p_* , is globally defined on $N \setminus D$ and non-negative.*

Proof: Let us denote by $D_{p_*}^k$ the diastasis of the metric $G_k = \varphi_k^* g_{FS}^k$ with respect to the point p_* . First of all observe that for $k \geq k_0$, the space $H^0(M, L^{\otimes k})$ is generated by $s_0^{\otimes j_0} \otimes \dots \otimes s_d^{\otimes j_d}$, $j_0 + \dots + j_d = k$ (cf. e.g. [6]). Thus, if conditions (C) and (D) hold, one can choose the map φ_k in such a way that $\varphi_k(p_*) = [1, 0, \dots, 0] \in \mathbb{C}P^{d_k}$. This implies that $D = \varphi_k^{-1}(Y_k)$, $\forall k \geq k_0$ where $Y_k = \mathbb{C}P^{d_k-1}$ is the hyperplane $Z_0 = 0$ in $\mathbb{C}P^{d_k}$. Since the diastasis of $(\mathbb{C}P^{d_k}, g_{FS}^k)$ with respect to $[1, 0, \dots, 0]$ is globally defined on $\mathbb{C}P^{d_k} \setminus Y_k$,

it follows by Proposition 1.1 that $D_{p_*}^k$ is globally defined on $N \setminus D$. By the very definition of diastasis in any coordinate system (Z_1, \dots, Z_n) centered in p_* we have: $D_{p_*}^k = \sum_{|i|, |j| \geq 0} a_{ij}(k) Z^i \bar{Z}^j$ where $a_{ij}(k)$ is a matrix depending on k such that $a_{j0}(k) = a_{0j}(k) = 0$. This is equivalent to

$$\frac{\partial D_{p_*}^k}{\partial Z^l}(p_*) = \frac{\partial D_{p_*}^k}{\partial \bar{Z}^m}(p_*) = 0, \quad (5)$$

for any multi-indices l, m and for any k . By the above mentioned theorem of Ruan the sequence $\frac{D_{p_*}^k}{k}$ C^∞ -converges to a Kähler potential Φ for the metric G (around the point p) which is then globally defined on $N \setminus D$. Since $D_{p_*}^k$ is non-negative on $N \setminus D$, being the restriction of the diastasis of the Fubini–Study metric on $\mathbb{C}P^{d_k}$, the potential Φ is non-negative on $N \setminus D$. Moreover property (5) goes to the limit, namely

$$\frac{\partial \Phi}{\partial Z^l}(p_*) = \frac{\partial \Phi}{\partial \bar{Z}^m}(p_*) = 0,$$

for any multi-indices l, m . Since (Z_1, \dots, Z_n) is an arbitrary coordinate system centered in p_* it follows, again by the definition of diastasis, that $\Phi = D_{p_*}$. \square

Remark 2.2 Observe that conditions (C) and (D) of Proposition 2.1 are satisfied by $(\mathbb{C}P^n, g_{FS})$. Indeed the polarization bundle L of $\mathbb{C}P^n$ is given by the hyperplane bundle and the space of holomorphic sections of $L^{\otimes k}$ can be identified by the homogeneous polynomials of degree k in the variables (Z_0, \dots, Z_N) (see e.g. [6]). A simple calculation shows that $Z_0^{j_0} \cdots Z_N^{j_N}, j_0 + \cdots + j_N = k$ is indeed an orthonormal basis for $H^0(\mathbb{C}P^n, L^{\otimes k})$ and hence conditions (C) and (D) are satisfied with respect to the point $p_* = [1, 0, \dots, 0]$. We also observe that conditions (C) and (D) hold for the product of two complex projective spaces (or a finite number of projective spaces) $N = \mathbb{C}P^r \times \mathbb{C}P^s, r + s = n$ endowed, as in the beginning of this section, with the sum of the Fubini–Study metrics on the two factors. In this case the polarization bundle L on N is given by the pull-back of the hyperplane bundle on $\mathbb{C}P^{r+s+s}$ via the Segre embedding. Moreover, a straightforward calculation shows that a orthonormal basis for the space $H^0(N, L^{\otimes k})$ is given by $Z_0^{j_0} \cdots Z_r^{j_r} W_0^{l_0} \cdots W_s^{l_s}, j_0 + \cdots + j_r = k, l_0 + \cdots + l_s = k$, and hence conditions (C) and (D) hold true with respect to the point $p_* = [1, 0, \dots, 0] \times [1, 0, \dots, 0]$.

We now discuss conditions (B), (B1) and (B2). Condition (B) is clearly satisfied by the above examples. Indeed in these cases the Bochner's coordinates with respect to the point p_* are given by the affine coordinates.

By Theorem 1.2, condition (B1) it is satisfied by any Kähler submanifold of the complex projective space.

We finally want to point out that condition (B2) in Theorem 1.6 is necessary. Let $N = \mathbb{C}^n / \mathbb{Z}^{2n}$ be the n -dimensional complex torus with the flat metric $G = \frac{i}{2} \sum_{j=1}^n |dz_j|^2$ coming from \mathbb{C}^n . This metric is polarized by the so called theta bundle. Let p_* be the origin in \mathbb{C}^n and let U be a fundamental domain containing p_* . The diastasis D_{p_*} of G with respect to p_* reads as $D_{p_*}(z, \bar{z}) = \sum_{j=1}^n |z_j|^2$ and the euclidean coordinates (z_1, \dots, z_n) are in fact Bochner's coordinates around p_* . Therefore U is an open dense subset of N for which all the properties of Theorem 1.6 are satisfied but, of course, U has finite euclidean volume. The same argument can be applied to any Riemann surface of genus $g \geq 2$ endowed with the hyperbolic metric which is polarized by the canonical bundle. Therefore these examples, having non-positive first Chern class show that the assumption (B2) is necessary.

3 Proof of the main results

Proof of Theorem 1.3:

Let p be a point in M such that $\varphi(p) = p_*$, where p_* is the point in N given by condition (A). Take Bochner's coordinates (z_1, \dots, z_m) in a neighborhood U of p which we take small enough to be contractible. Since the Kähler metric g is Einstein with (constant) scalar curvature s then: $\rho_\omega = \lambda \omega$ where λ is the Einstein constant, i.e. $\lambda = \frac{s}{2m}$, and ρ_ω is the *Ricci form*. If $\omega = \frac{i}{2} \sum_{j=1}^m g_{j\bar{k}} dz_j \wedge d\bar{z}_k$ then $\rho_\omega = -i\partial\bar{\partial} \log \det g_{j\bar{k}}$ is the local expression of its *Ricci form*.

Thus the volume form of (M, g) reads on U as:

$$\frac{\omega^m}{m!} = \frac{i^m}{2^m} e^{-\frac{\lambda}{2} D_p + F + \bar{F}} dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_m \wedge d\bar{z}_m, \quad (6)$$

where F is a holomorphic function on U and $D_p = \varphi^{-1}(D_{p_*})$ is the diastasis on p (cfr. Proposition 1.1).

We claim that $F + \bar{F} = 0$. Indeed, observe that

$$\frac{\omega^m}{m!} = \frac{i^m}{2^m} \det\left(\frac{\partial^2 D_p}{\partial z_\alpha \partial \bar{z}_\beta}\right) dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_m \wedge d\bar{z}_m.$$

By the very definition of Bochner's coordinates it is easy to check that the expansion of $\log \det(\frac{\partial^2 D_p}{\partial z_\alpha \partial \bar{z}_\beta})$ in the (z, \bar{z}) -coordinates contains only mixed terms (i.e. of the form $z^j \bar{z}^k, j \neq 0, k \neq 0$). On the other hand by formula (6)

$$-\frac{\lambda}{2}D_p + F + \bar{F} = \log \det(\frac{\partial^2 D_p}{\partial z_\alpha \partial \bar{z}_\beta}).$$

Again by the definition of the Bochner's coordinates this forces $F + \bar{F}$ to be zero, proving our claim.

By Theorem 1.2 there exist Bochner's coordinates (Z_1, \dots, Z_n) in a neighborhood of p_* satisfying (2). Moreover, by condition (B) this coordinates are globally defined on X . Hence, by formula (6) (with $F + \bar{F} = 0$), the m -forms $\frac{\Omega^m}{m!}$ and $e^{-\frac{\lambda}{2}D_{p_*}} dZ_1 \wedge d\bar{Z}_1 \wedge \dots \wedge dZ_m \wedge d\bar{Z}_m$ globally defined on X agree on the open set $\varphi(U)$. Since they are real analytic they must agree on the connected open set $\hat{M} = \varphi(M) \cap X$, i.e.

$$\frac{\Omega^m}{m!} = \frac{i^m}{2^m} e^{-\frac{\lambda}{2}D_{p_*}} dZ_1 \wedge d\bar{Z}_1 \wedge \dots \wedge dZ_m \wedge d\bar{Z}_m. \quad (7)$$

Since $\frac{\Omega^m}{m!}$ is a volume form on \hat{M} we deduce that the restriction of the projection map

$$\pi : X \cong \mathbb{C}^n \rightarrow \mathbb{C}^m : (Z_1, \dots, Z_n) \mapsto (Z_1, \dots, Z_m)$$

to \hat{M} is open. Since it is also algebraic (because the biholomorphism between X and \mathbb{C}^n is algebraic), its image contains a Zariski open subset of \mathbb{C}^m (see Theorem 13.2 in [2]), hence its euclidean volume, $\text{vol}_{\text{eucl}}(\pi(\hat{M}))$, has to be infinite.

Suppose now that the scalar curvature of g is non-positive. By formula (7) and by the fact that D_{p_*} is non-negative, we get $\text{vol}(\hat{M}, g) \geq \text{vol}_{\text{eucl}}(\pi(\hat{M}))$ which is the desired contradiction, being the volume of M (and hence that of \hat{M}) finite. \square

Proof of Theorem 1.4: Since the metric G is K-E, working as in Theorem 1.3 one gets that

$$\frac{\Omega^n}{n!} = \frac{i^n}{2^n} e^{-\frac{\lambda}{2}D_{p_*}} dZ_1 \wedge d\bar{Z}_1 \wedge \dots \wedge dZ_n \wedge d\bar{Z}_n, \quad (8)$$

on a contractible open set U where the Bochner's coordinates (Z_1, \dots, Z_n) are defined. Let (f_1, \dots, f_n) be holomorphic functions on X which extend (Z_1, \dots, Z_n) (given by condition (B1)). Then the n -forms $\frac{\Omega^n}{n!}$ and

$e^{-\frac{\lambda}{2}D_{p*}}df_1 \wedge d\bar{f}_1 \wedge \dots \wedge df_n \wedge d\bar{f}_n$ globally defined on X agree on the open set U . Since they are real analytic they must agree on X , i.e.

$$\frac{\Omega^n}{n!} = \frac{i^n}{2^n} e^{-\frac{\lambda}{2}D_{p*}}df_1 \wedge d\bar{f}_1 \wedge \dots \wedge df_n \wedge d\bar{f}_n. \quad (9)$$

As a consequence the map

$$\pi : X \cong \mathbb{C}^n \rightarrow \mathbb{C}^n : x \mapsto (f_1(x), \dots, f_n(x))$$

is open. Moreover, since it is holomorphic it is also surjective and hence $\text{vol}_{eucl}(\pi(X))$ is infinite.

Suppose that the scalar curvature of G is non-positive. By formula (9) and by the fact that D_{p*} is non-negative, one gets that $\text{vol}(X) \geq \text{vol}_{eucl}(\pi(X))$ which is the desired contradiction, being the volume of X finite. \square

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